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*Review*
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# Joseph Fourier's Enduring Legacy: From Mathematical Physics to Modern Spectroscopy and Biopharmaceutical Innovation

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**Abstract:** In this article, we highlight the fascinating life of Joseph Fourier, his remarkable heat equation, the invention of the famous Fourier series, the Fourier transform, and other breakthroughs that used his discoveries as tools. Fourier left his mark not only on Mathematical Physics and Pure and Applied Mathematics, but also had a major impact on his theories in Economy, Signal and image processing, Quantum Computing, Cryptography, Operations Research, and many more areas of Applied Sciences. Almost all vital areas, like the financial sector, Medical Science, Instrumentation, and Industrial, are highly indebted to the fantastic vista laid by Fourier, without which we could not have reached the present-day high-technology era. Currently, research on Heritage and Conservation Science has gained momentum due to extensive work employing ATR-FTIR spectroscopy alongside GC-MS. In recent times, "ATR-FTIR spectroscopy and spectroscopic imaging processes" have been employed for the analysis and production of biopharmaceuticals, drugs derived from biological organisms for the treatment of diseases such as arthritis, diabetes, and certain cancers. Further, it is also being viewed as a potential process for producing monoclonal antibodies to support the ongoing fight against SARS-CoV-2.

**Keywords:** Fourier transformation; signal processing; cryptography; operations research; FTIR; medical science.

## 1. Introduction

Regarding the preliminary research by d'Alembert and Euler, they were unable to add factors to the expansion, which rendered it unsuitable for use. They both persuaded that an arbitrary and discontinuous function could never be resolved in a series of this kind, and it does not seem that anyone had developed a constant in cosines of multiple arcs, the first problem which I had to solve in the theory of heat." -Joseph Fourier [1]

From the 18$^{th}$ century onward, Long strides were taken in the mathematical treatment of issues in "the physical sciences: heat, sound, light, fluid dynamics, elasticity, electricity, and magnetism". Numerous new findings in both domains were made possible by the intricate interaction between mathematics and its applications. The problem that started the whole development was not simple and did not come from physics but from music. It was Pythagoras who was impressed, got encouraged in music, which influenced him to try to do some experiments. His tremendous discovery was that two notes make a pleasant sound, and melodious, if corresponding string lengths are in the numerical ratios such as 2:1, 3:2, or 4:3, and or reverse [2, 3]. After the passage of more than two millennia, it was possible for mathematics to describe why such proportions are naturally formed in elastic strings.

The vibrations in continuous systems were based on "the law of elasticity" by Robert Hook (1635 − 1703) in the year 1660 and "Newton's second law of motion" in the year 1687, which were basic to the Mathematical Model of vibrating systems. "Jacob Bernoulli (1655 − 1705)", brother of "Johann Bernoulli (1667 − 1748)", solved the catenary problem that was proposed by his brother Jacob in the year 1691. "Brook Taylor (1685 − 1731)" was inspired by music and discussed vibrating strings in his own book "Methodus incrementorum directa et inversa" in the year 1715. When "Daniel Bernoulli (1700 − 1782)" lived in St. Petersburg, He analysed how an elastic line might look under "the action of forces" and also showed an endless number of harmonic vibrations superposed, which are formed by the motion of a string. Further, without mass, the free vibrations of an elastic string were examined by "Euler (1707-1783)" [4]. In the same period, d'Alembert (French mathematician) (1717 − 1783) developed a method to resolve a system of "linear differential equations regarding the solution of a vibrating string". In this framework, Daniel Bernoulli proved that the solution to the free vibration of a string is a trigonometric series. Such discussion went on over a few decades among Euler, d'Alembert, and Daniel Bernoulli.

In 1714, Brook Taylor (an English mathematician) noticed the matter and determined the fundamental vibration frequency of the instrument string in terms of string length, tension, and density. The Greeks were aware that an instrument string may generate a wide range of sonic notations based on the placement of nodes, or the fixed rest-points, as the musical route is known in modern parlance. Frequency is the quantity of vibrational cycles that a moving item completes in one second, and as the string fluctuates faster, more frequently, and produces a higher note. Only the terminal points are at rest in the case of fundamental frequency. If the string contains a node in the middle, it will produce nodes exactly twice as often (see Appendix A for details). Further, as the number of nodes increases, higher frequencies or more vibrations occur, which are called overtones.

However, the actual matter is quite different and not an easy task. Multiple vibrations produced by violin strings are not the normal modes, which was shown by d'Alembert in the year 1746. He had also shown that the geometrical figure of the wave at a time $t = 0$ is arbitrary. He considered a string having length l, that stretched out from “(0, 0) to (l, 0)” along axis-X, and at “time t = 0, point (x, 0)” is shifted to y(x, t) along the direction of Y, as shown in Figure 1.

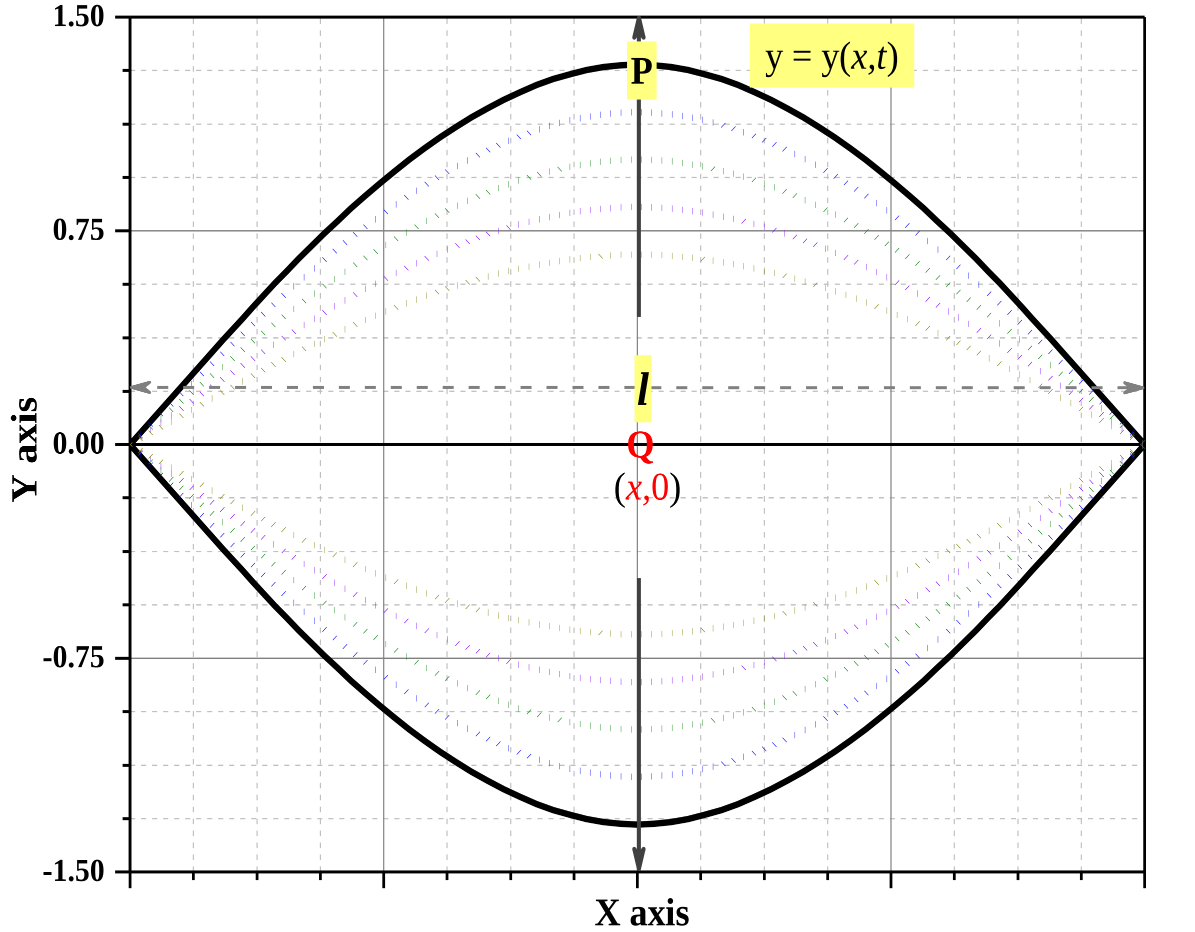


**Figure 1.** Movement of a vibrating string having length *l* attached at two ending points (*0*, *0*) and (*l*, *0*). The point (*x*, *0*) is shifted to *y*(*x*, *t*) in the direction of the *Y* axis, at time *t* = *0*.

Here, the function $y(x, t)$ is dependent on both $x$ and $t$. Now, keeping x fixed, $f(t) = y(x, t)$ is an arbitrary function. Considering this, the d’Alembert wave equation is:

$$y_{tt} = c^2 y_{xx} \qquad (1)$$

It can be shown that the (1.1) satisfies the two states, “$y(0, t) = 0$ and $y(l, t) = 0$, $\forall t$”, where $c = 299792458$ m/s, velocity of light in the case for electromagnetic problem (Figure 2) and $c = 343$ m/s, velocity of sound in the case for acoustics and elasticity mechanics (Figure 3), considering the arbitrary function $\eta(x) = \sin x$ for both the cases [5].

Solving equation (2), the general solution (general) is:

$$y(x, t) = \frac{1}{2}[\eta(x + ct) + \mathbb{O}(x - ct)] \qquad (2)$$

The 1st boundary condition,$y(0, t) = 0$, shows that $\eta$ is odd, and the second boundary condition,$y(l, t) = 0$, shows that $\eta$ has a period of $2l$.

In this case, constant c is related to the strictness of the string, and the functions $\eta$ and $\mathbb{O}$ in one variable are arbitrary. The two arbitrary functions $\eta$ and $\mathbb{O}$has have significant physical meaning. The function $\eta$ represents the forward wave, which travels in the $X$-axis with speed $c$ in the negative or opposite direction to that of $\mathbb{O}$, and the function $\mathbb{O}$ represents the backward wave that moves along in the positive direction of the $X$-axis with speed $c$. Therefore, this solution represents the superposition of these two waves, which travel in opposite directions, resulting in a complex waveform [6].

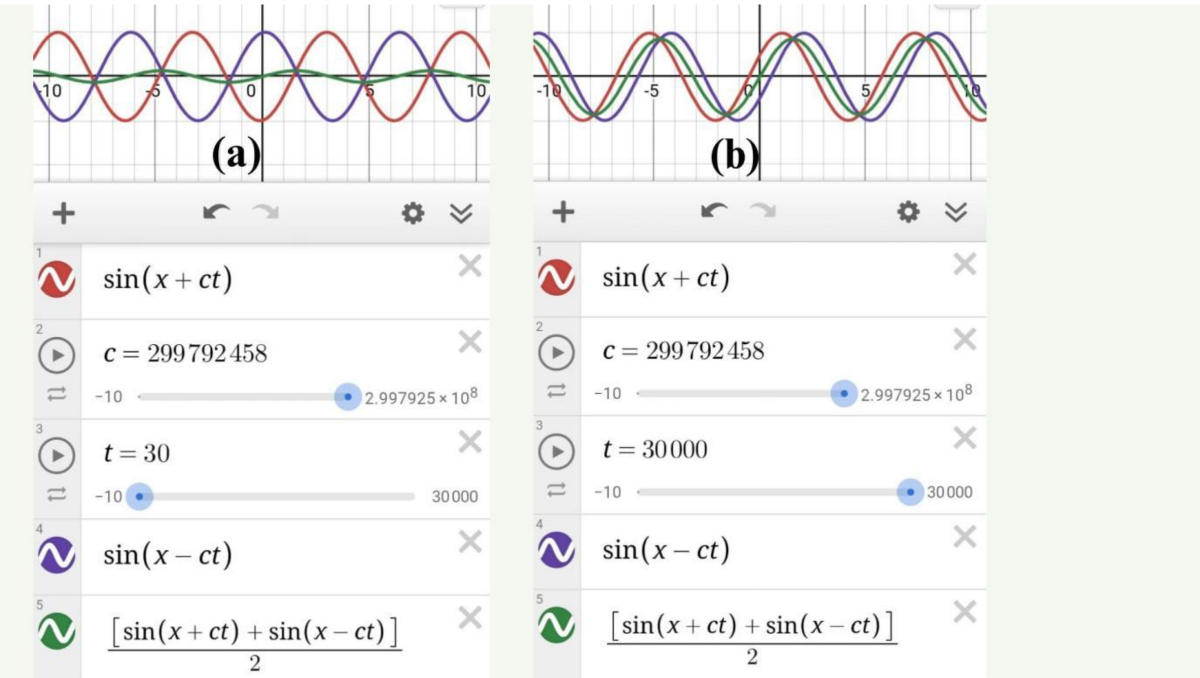


**Figure 2.**Graphical representation of $y(x,t)$, equation (2), for electromagnetic problem **(a)**$t$ = 30 s; **(b)**$t$ = 30,000 s.

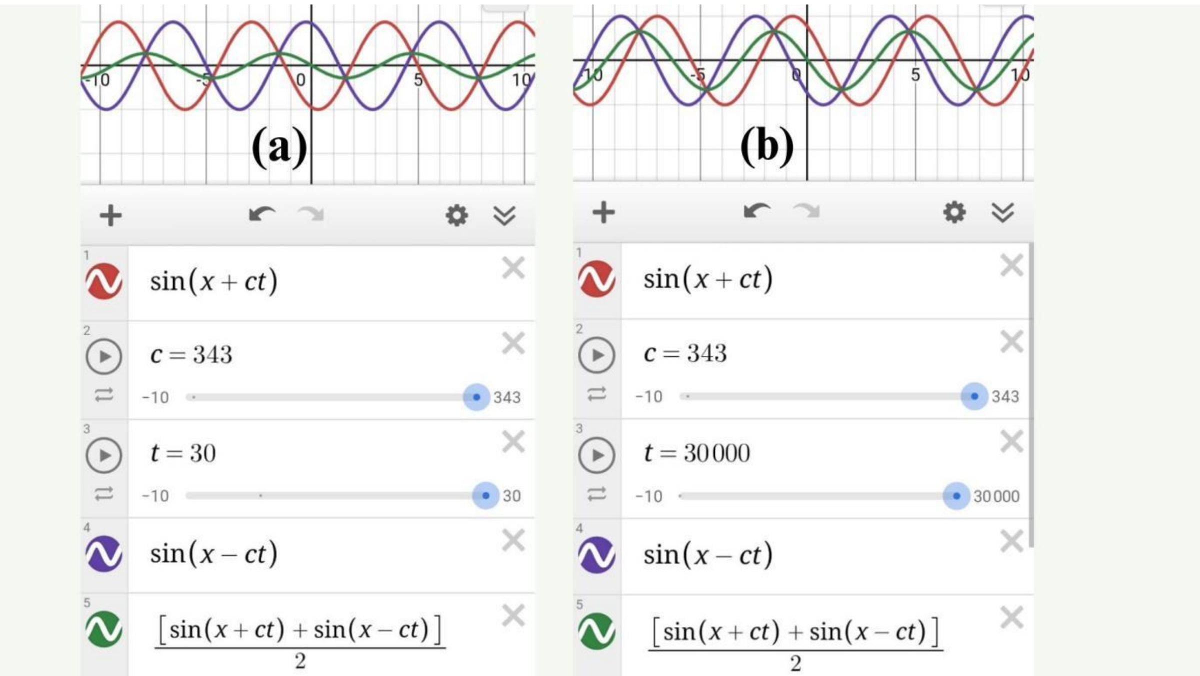


**Figure 3.**The graphical representation of $y(x,t)$, equation (2), for acoustics and elasticity Mechanics is given for **(a)**$t$ = 30 s; **(b)**$t$ = 30,000 s.

In 1822, Fourier solved the problem completely in his book "The Analytical Theory of Heat," in which it is shown that the heat conduction in three-dimensional (3-D) space is represented by the differential equation (3), where $u^*$(temperature distribution function) = $u^{**}(\boldsymbol{r}, t)$

$$\frac{\partial u^{**}}{\partial t} = \kappa\left[\frac{\partial^2 u^{**}}{\partial x^2} + \frac{\partial^2 u^{**}}{\partial y^2} + \frac{\partial^2 u^{**}}{\partial z^2}\right] \quad (3)$$

where "$\boldsymbol{r}$ is the position vector, $\kappa$ is the constant of proportionality (= $k/\sigma\rho$, $k$ = heat conductivity, $\sigma$ = specific heat, $\rho$ = density)", called diffusivity constant [1]. He also mentioned in his book "any periodic function can be represented as an infinite sum of sine and cosine in trigonometric form, which is not valid unless it is a bounded variation or piecewise continuous" [7]. One dimensionally, one way to spell it is as:

$$y = \frac{l_0}{2}(l_1 \cos x + l_1' \sin x) + (l_2 \cos 2x + l_2' \sin 2x) + \cdots \tag{4}$$

The above equation (4) is known as the Fourier series, where $l_0, l_1, l_2, \ldots$ and $l_0', l_1', l_2', \ldots$ so on, are called Fourier coefficients. It is always periodic in a bounded region. A groundbreaking operator, the Fourier Transformation (FT) was established by J. Fourier in 1807 (manuscript submitted to the Institute of France) using the series (4), although it was published in his book "Analytic Theory of Heat", 1822 [8–10] (calculation details of FT from Fourier series in Appendix B).

Digital imaging, Signal processing, and Computer vision are considered vital components of the digital industry, medical, security, forensic, and many other applications [11]. Incorporation of FT processing has played an immense role in developing the above area. Properties of FT have been discussed in Section 6, for the purpose of image and signal processing.

It has been shown that, in the case of unbounded regions, despite certain limitations for $t$, the Fourier transform for "the function $\eta(t)$" we have:

$$\mathcal{F}(\eta(t)) = \int_{-\infty}^{\infty} \eta(t) e^{-i\chi t} dt \tag{5}$$

Separating the real and imaginary parts of the right-hand side of equation (5), we have:

$$\mathcal{F}(\eta(t)) = \Re\left(\mathcal{F}(\eta(t))\right) + i\Im\left(\mathcal{F}(\eta(t))\right) \tag{6}$$

whereas $i = \sqrt{-1}$ and equate with the separation of the right-hand side. Hence:

$$\Re\left(\mathcal{F}(\eta(t))\right) = \int_0^{\infty} \eta(t) \cos \chi t \, dt \tag{7}$$

$$\Im\left(\mathcal{F}(\eta(t))\right) = -\int_0^{\infty} \eta(t) \sin \chi t \, dt \tag{8}$$

Here, it is obvious that $\Re(F(\eta(t)))$ is an even function and $\Im(F(\eta(t)))$ is an odd function. These equations are of the sine and cosine transform; they can be inverted to two different formulae for $\eta(t)$. Hence, we have:

$$\eta(t) = \frac{2}{\pi}\int_0^{\infty} \Re\left(\mathcal{F}(\eta(t))\right) \cos \chi t d\chi = -\frac{2}{\pi}\int_0^{\infty} \Im\left(\mathcal{F}(\eta(t))\right) \sin \chi t d\chi \quad (t > 0) \tag{9}$$

$$\int_0^{\infty} \Re\left(\mathcal{F}(\eta(t))\right) \cos \chi t d\chi = -\int_0^{\infty} \Im\left(\mathcal{F}(\eta(t))\right) \sin \chi t d\chi \tag{10}$$

Using equation 6 and the inverse Fourier transformation, we have:

$$\eta(t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \left\{\Re\left(\mathcal{F}(\eta(t))\right) cos\, \chi t - \Im\left(\mathcal{F}(\eta(t))\right) \sin \chi t\right\} d\chi + i \int_{-\infty}^{\infty} \left\{\Re\left(\mathcal{F}(\eta(t))\right) sin\, \chi t + \Im\left(\mathcal{F}(\eta(t))\right) \cos \chi t\right\} d\chi\right] \tag{11}$$

If the term [$\Re(F\ (\eta(t)))\ sin\ \chi t + \Im(F\ (\eta(t)))\ cos\ \chi t$] is odd, then the imaginary part of equation 11 is zero. For t > 0, using equation 9, the two terms of the real part of equation 11 are equal, and we obtain a non-zero result. In the case of $t<0$, it is clear that the real part of equation 11 vanishes [12]. Fourier Series [FS] and Fourier Transform [FT] have brought about a revolutionary new dimension for understanding physical phenomena and opened a new era of science. Almost every branch of science has been enriched by this method. As a consequence, a host of fantastic applications emerged involving FT in Cryptography, Quantum cryptography, Economy, Economics, and Business cycles, Signal and image processing, Operations Research, and many more in Chemistry, Bio-sciences. FT is also used to study the stability of "linear differential equations" [13].

## 2. Short Biography

"Jean-Baptiste Joseph Fourier (1768-1830)" was an eminent French mathematician, Egyptologist, and introducer of new sparks in pure and applied Mathematics and the field of Egyptology. A small town, Auxerre in France, situated on the Yonne River's bank, depicting diversity in "cultural, religious, political and educational traditions", where he was born on 21st March, 1768. His mother worked as a housewife, while in his profession, "Joseph's father was a tailor from Auxerre". He lost both his parents at the age of nine or ten. To learn Latin and French, he first enrolled in Pallais Elementary School. When he was thirteen years old, he could successfully finish a comprehensive study of the total six volumes of "Etienne Bezout's (1730–1783) Cours de mathématique". In the year 1783, he achieved 1st place in his analysis of "C. Bossut's (1730-1814) Mécanique". At an early age, Joseph presented his research at "The Academie Royale des Sciences", where he presented a study on algebraic equations. In 1787, he thought of becoming a priest and wanted to take training for the same. However, he changed his decision after coming in contact with the professor of mathematics, C L Bonard at Auxerre, and got admitted to "Ecole Royale Militaire" to learn science, including mathematics. He started his teaching profession to teach Mathematics at the Auxerre, France, "Benedictine College's Ecole Royale Militaire" in 1790, where he also studied. He played a major role in promoting the French Revolution in 1789-1794 and was twice detained on the Committee for Public Safety's orders due to his alleged revolutionary, terrorist activity during the Auxerre Revolution in 1793-1794. Fourier engaged in a local political revolution, which directly caused two incarcerations, for which he endured excruciating anguish. Next, he decided to escape from all political trouble and wanted to fight for freedom and life. This was a turning point for him to pursue a great "research career in mathematics and physics". In this field, he marked his identity as a special person in the later part of his life. Joseph was selected in 1794 to attend "The Ecole Normale in Paris", which began as a prototype institution for teacher training in France in 1795. He had a unique opportunity to interact with some of the time's well-known mathematicians, such as Gaspard Monge, J. L. Lagrange, and P. S. Laplace, while attending teacher training school (1746–1818). He began working as "an assistant lecturer" at "The Ecole Polytechnique" in September 1795 to assist with the instruction of "Monge and Lagrange". In 1798, he was selected as scientific advisor for the Egyptian expedition led by Napoleon Bonaparte (1769–1821), which was intended to free the dissatisfied people of Egypt from severe political and social issues and extend his hand to grant them access to all the advantages of European civilization. During this occupation in Egypt, he became "permanent secretary" in "the Institute d'Egypt", established by Napoleon, in Cairo with Monge as its President". Due to this stance, Fourier decided to write the Description "de l'Egypte (Description of Egypt)", which was partially edited by Napoleon himself and had a beautiful introduction. Napoleon made his way back to Paris in 1799 to seize total control of France, and he soon proclaimed himself "Emperor of France". In 1801, Fourier returned to "Paris" to restart his duties as "Ecole Polytechnique Professor of Analysis". Fourier was appointed "Prefect of Isere in Grenoble" after his departure for Egypt in 1802, taking on a wide range of important and complex administrative responsibilities.

In 1807, Fourier submitted his classical memoir on heat conduction, entitled "On the propagation of heat in solid bodies," to the Paris Academy of Sciences for a research award. In his article, he developed the trigonometric series of expansion of functions without proper justification, which led to rejection for the prize by the committee formed by Monge, S. L. Lacroix (1765–1843), Lagrange, A. M. Legendre (1752–1833), and Laplace. Even though the Academy encouraged Fourier to submit his work again for a grand prize to be given in 1812 because of the significance of the subject matter and innovation, however on the contrary, his most rivalry "J. B. Biot (1774–1862)" and "S. D. Poisson (1781–1840)" voiced grave worry over significant Objections to Fourier's work regarding his presentation of "The theory of heat in solids" in both mathematical and physical terms.

Despite several disagreements, Fourier continued his research work with his extraordinary talent and courage. In 1811, he provided an updated version of his memoir, as well as additional information on "the cooling of infinite solids, terrestrial and radiant heat," a contrast between his theory and practical findings, and computations for "the flow of heat in fluids." He received the Golden Prize for his outstanding effort in 1812. Fourier was unanimously chosen to join "The Academie des Sciences" in 1817. The Academie des Sciences' "J.B. Delambre (1749–1822)" was "the division's permanent secretary for mathematics and physics". He passed away in 1822, and this post was assigned to Fourier till the end of his life in 1830. As soon as he was appointed Secretary, the Académie released "The Analytical Theory of Heat" in 1822. Its foundation was "Isaac Newton's (1642-1727)" equation of cooling, which states "the passage of heat between two nearby molecules is proportionate to a very tiny temperature difference".

After Laplace and Jean D'Alembert (1717–1783), Fourier was elected into the French Academy of Literature. Upon Laplace's passing, Fourier was also chosen to lead "The Council de perfectionnement of The École Polytechnique" in the year 1827. Due to his exceptional input to the mathematical sciences, Fourier was chosen as an honorary member by a number of international scientific associations, including "The Royal Society of London and The Royal Swedish Academy of Sciences". During his Egyptian expedition as scientific advisor, he possibly contracted something that led to the display of some unusual symptoms at the advanced period of his life. Fourier experienced a heart attack in the first few days of May 1830, and his condition progressively worsened until he passed away on May 16 at the age of 62. Because of his exceptional work as the Cairo Institute's Permanent Secretary and the compilation of the ground-breaking Description de l'Egypte, his gravestone was decorated with an Egyptian pattern. [1].

## 3. Cryptography

The art of converting information into a form that is unintelligible to an unintended recipient is called cryptography, and the person who designs the code or key to convert the source text into a different form for the destination is called a cryptographer [14]. The art of breaking the aforementioned system, carried out by a cryptanalyst who breaks the design of the key/code, is called cryptanalysis. The information to be converted to another form by the encryption process is called Plain text, or message text, and after encryption using a key or secret code, the resulting form is called ciphertext. If anyone knows the key or code for the encryption, after getting the ciphertext, they can decrypt it to get back the original message. Therefore, the key or code is vital to keep the information secret in an encrypted form. Ciphertext is sent to the receiver through a public channel accessible to all of us, but the key is not. So the whole process of encryption and sending it via a public or insecure channel to the destination or receiver is a challenging part of this system. This encryption and decryption process, when used with a public channel, is called a cipher or cryptosystem. Therefore, a cryptosystem helps the designer communicate with a person or group of persons over a channel that isn't secure, so that a rival, bad guy, or hacker (Oscar) cannot understand what is being said. Figure 4 depicts the communication between two persons, Alice and Bob, over a public channel.

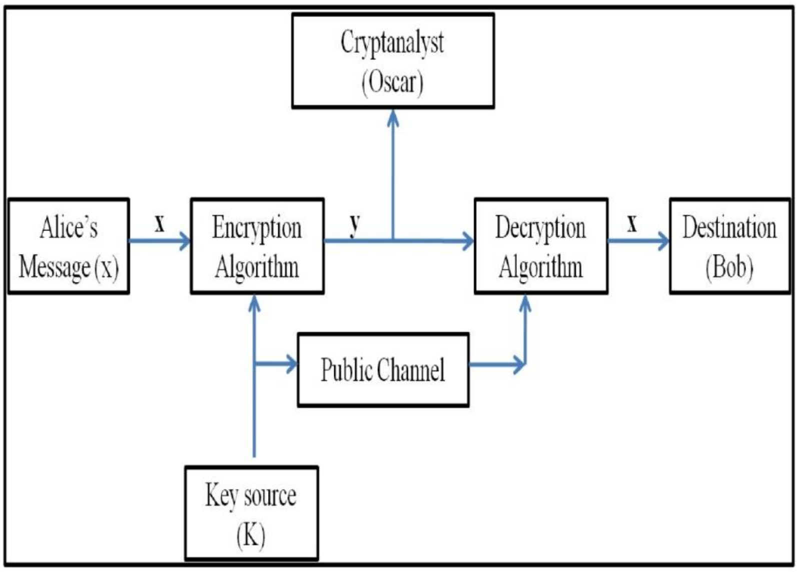


**Figure 4.** Block diagram of the message transmission process through encryption and decryption over a public Channel.

The two primary types of cryptosystems are [15]:

### *3.1. Symmetric key cryptosystem.*

A cryptosystem is called *a* "symmetric key cryptosystem"[16] if *the* same key is used for *the* encryption process as well as *the* decryption process.

### *3.2. Asymmetric key cryptosystem or public key cryptosystem.*

"A cryptosystem is called an asymmetric key cryptosystem or public key cryptosystem" [17] if the same key is not used for the encryption process as well as the decryption process, but a pair of keys, public key (that may be known to others) and private key (which is never known to others except the designer or owner) is used in this cryptosystem. "Public key cryptography" was invented by Diffie and Hellman [15], and Merkle [18] radically changed the approach to key distribution.

(a) Presently, the use of biometrics is gaining momentum in almost every sphere of government and the private sector to verify personal identification. It is the human traits, such as face, fingerprints, irises, etc., and therefore they represent unique characteristics for individuals. It provides access to the system for verifying sensitive personal information, so its encryption process should be robust, secure, and protect privacy. Sufficient care is required to process biometric data; otherwise, it may be misused by any third party or dishonest person. Attack on this encryption process is so much more secure, invulnerable [19], and due to the distinct. In symmetric key cryptography, the cryptosystem depends heavily on the key characteristics of individual biological trials; it produces a correct and strong biometric key [20]. In symmetric key cryptography, the cryptosystem depends heavily on the key [21] ("using the same key for encryption and decryption"), which is used for both encryption and decryption. So, symmetric key cryptosystems face problems in real-time applications for delivering or distributing the key used in the encryption process. In order to overcome this problem, an asymmetric key cryptosystem was introduced [22]. Using "Fourier domain cryptography or double random phase encoding (DRPE)" [23, 24], phase truncated Fourier transform [25, 26], and two Fourier transformations [27], an asymmetric cryptosystem was proposed to encrypt and decrypt the biometric signature in the following way.

A biometric signature is used as plain text ($b_1$), and two other biometric signatures ($b_2$ and $b_3$) are required for the next steps. Using DRPE, $b_2$ and $b_3$ are encrypted, and the corresponding ciphers are $c_2$ and $c_3$, which are multiplied in the subsequent step after multiplying by the exponential form. Two private phases, key $PK_2$ and $PK_1$, are reserved by the phase reservation operator and are essentially used as decryption keys in the decryption algorithm. "Here, PT is the phase truncation operator; FT and IFT are used as the Fourier transformation and inverse Fourier transformation, respectively," which is shown in the following flow diagram (Figure 5) [20]. So, here, the encryption of b1 is privacy-protected, and two decryption keys are generated by the phase reservation operator (PR). "The encryption and decryption keys are not identical." That's an asymmetric-key cryptosystem for privacy-protected, encrypted biometric signatures using two Fourier transformations [27, 28] and DRPE (double-random-phase encryption) [23, 20].

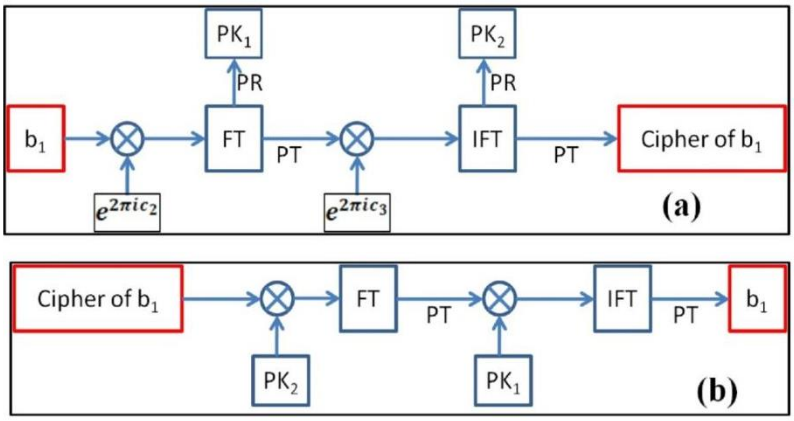


**Figure 5.** Asymmetric cryptosystem: **(i)** Encryption; **(ii)** Decryption.

(b) To provide a secure cryptographic scheme, the role of symmetric key encryption as "AES (Advanced Encryption Standard)" and "DES (Data Encryption Standard)" [29] is extremely vital. The encryption scheme above is very complex, which makes hardware and software implementation difficult due to the high storage requirements. It is surely better if an encryption scheme is available that requires minimal storage while maintaining strong security. Using the Fourier series, the desired type of encryption is available, providing strong security and being implementable on small devices that do not require large storage capacity.

(c) On the other hand, there is secrecy in transferring data, which gets transformed into data by encrypting using the Moore machine [30] and the Fourier transform. The input of a Moore machine is done by the Fourier transformation of a function that is piecewise continuous with exponential order and messages that are used in the expansion of that function.

## 4. Quantum Computing

"If the computers that you build are quantum
Then spies everywhere will all want'em
Our codes will all fail,
And they'll read our email,
Till we get crypto that's quantum, and daunt'em." —-Jennifer and Peter Shor

"Quantum mechanics is a guideline or set of mathematical instructions for the construction of physical concepts to describe the nature of atoms and waves remarkably on an atomic and subatomic scale." High-speed computation processes can be easily solved by quantum computing mechanisms. A classical computer that works using only 0 or 1 at a time to perform a given task cannot effectively model the evolution of a quantum system. To overcome this problem, Richard Feynman proposed a basic model at MIT in 1981. This model can simulate such a problem in exponential time rather than the classical time of a computer. This model gave birth to the quantum computer. One of the first quantum algorithms, Grover's algorithm, was solved by the 2-qubit quantum computer, which was first built in 1998 [31]. In 2017, IBM was the first to bring a usable commercial quantum computer. The latest quantum computer with 127 qubits was first introduced by IBM in November 2021.

A computation is a task performed by any method using a device such as a computer. Nowadays, we use computers that operate using 0s and 1s, a classical method. On the other hand, in quantum computer we have the advantage to use zero, one and superposition of zero and one i.e. "$p|0\rangle + q|1\rangle$, where p and q are complex numbers such that $|p|^2 + |q|^2 = 1$". Here, "$|0\rangle$ and $|1\rangle$ are two quantum states" such that quantum bit *0* occurs with the probability $|p|^2$ and that for quantum bit 1 is $|q|^2$ where $p|0\rangle + q|1\rangle$ is called qubit. These two states are vectors in a two-dimensional complex vector space with $|0\rangle$ and $|1\rangle$, and are an orthonormal basis [32]. For multiple qubit we can write like "$p_1|00\rangle + p_2|01\rangle + p_3|10\rangle + p_4|11\rangle$ and $q_1|000\rangle + q_2|001\rangle + q_3|010\rangle + q_4|011\rangle + q_5|100\rangle + q_6\,|110\rangle + q_7\,|101\rangle + q_8\,|111\rangle$", where $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$ and $q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 + q_7^2 + q_8^2 = 1$ for formal and later case; respectively where "$p_1$, $p_2$, $p_3$, $p_4$, $q_1$, $q_2$, $q_3$, $q_4$, $q_5$, $q_6$, $q_7$, $q_8$" are complex numbers.

Classical computers work with bits 0 and 1 only; as a result, for a three-bit digit, that computer needs eight steps. However, quantum computers perform this task in one step because they contain eight switches that can perform any one of eight tasks in a single attempt. In the following text, we will discuss using the Fourier transform and how this fact helps make an efficient algorithm to solve the factoring problem, which has been carried out safely for hundreds of years without an efficient algorithm.

### *4.1. RSA scheme.*

Today's life is increasingly driven by digital platforms, where our main challenge is to transfer and store data while ensuring four essential services: Confidentiality, Authentication, Data Integrity, and Nonrepudiation. For these purposes, encryption is the main tool to maintain the security that has been maintained for over four decades by the "RSA encryption system," presented in 1978 by "Rivest, Shamir and Adleman." This encryption scheme is given as follows:

1. Let two primes "$u$ and $v$ and $\mu = u.v$"
2. Find "$\phi(\mu) = (u-1)(v-1)$"
3. Choose "$N > 1$ an integer such that gcd $(N, \phi(\mu)) = 1$"
4. Compute "$d' = N^{-1} \bmod \phi(\mu)$"
5. "$\mu$, N, $d'$ "

Here $\mu$ and N are the "public key," and d′ is the "private key". Here, hardness is to factorize $\mu$; if anyone can find factors $u$ and $v$ of $\mu$, they can compute d′ and decrypt all the encrypted messages using this scheme [29]. If $\mu$ is of 200 digits, a classical computer needs 2000 years to factorize it. But in a quantum computer using the quantum factoring algorithm that was introduced by Peter W. Shor [33], anyone can factor N in a reasonable amount of time. A quantum algorithm works by a quantum gate that is accepted only if the corresponding matrix in the truth table is unitary, that is, its inverse or conjugate transpose. In construction, a Discrete Fourier transformation is used on a quantum computer to construct a particular unitary. After that, the quantum Fourier transformation is used, which is exactly the same as the discrete Fourier transformation but appears differently due to its presentation in conventional notation [33].

*4.2. Discrete Fourier transformation (DFT).*

Discrete FT (Df) takes a vector of complex numbers as input and generates a vector in complex form as output that is the sum of complex exponentials, and the vector in complex form looks like $(x_0, x_1, \ldots, x_{\alpha-1})$ where fixed parameter α represents the length. This transformation is defined by:

$$y_k = \frac{1}{\sqrt{\alpha}} \sum_{j=0}^{\alpha-1} x_j e^{\frac{2\pi ijk}{\alpha}} \tag{12}$$

where $k = 0, 1, \ldots, \alpha - 1$.

This transform provides us with the discrete Fourier transform matrix (Appendix C) [33]. This is so difficult to solve due to the arithmetic complexity for large $\alpha$. Cooley and Tukey (1965) published an algorithm, which is a much faster and more effective algorithm called Fast FT (FFT) [34]. The approach of this algorithm is the divide and conquer method to diminish the computational complexity of DFT from $O(\alpha)$ to $O(\alpha log_2\alpha)$. FFT is used in many areas of science, including encryption models such as Multi-biometric Image Encryption [35] and Verifying Signatures Online [36], and many applications have been developed using this transformation [37, 38].

*4.3. Quantum Fourier transformation.*

Quantum Fourier transformation is the same as DFT (Discrete Fourier transformation), but the notation of representation is quite different. The quantum Fourier transforms is defined on orthonormal basis $|0\rangle, \ldots, |\beta-1\rangle$ in such a following action:

$$|j\rangle \rightarrow \frac{1}{\sqrt{\beta}} \sum_{k=0}^{\beta-1} e^{\frac{2\pi ijk}{\beta}} |k\rangle \tag{13}$$

Similarly, for the arbitrary state action is

$$\sum_{j=0}^{\beta-1} x_j|j\rangle \rightarrow \sum_{k=0}^{\beta-1} y_j|k\rangle \tag{14}$$

Here, amplitudes $y_k$ represent the DFT of the amplitudes $x_j$. Transformation is unitary, and so it may be implemented on a quantum computer [32].

*4.4. Shor's algorithm.*

There is a fast, efficient algorithm using the idea of the product of two prime numbers, but conversely, there does not exist an efficient algorithm to get back those "two prime numbers" in a reasonable length of time. But "Shor's algorithm" is the remarkable weapon to factor the product μ of two primes [33]. This is preceded by resolving a related issue that choosing $\zeta \leq \mu$, and ζ is relatively prime to μ, and finding the order r of ζ ( mod μ ) using Euclid's algorithm. But in quantum Shor's algorithm, we take random $\zeta < \mu$ with M-bit register in the state $|0\rangle$ where $q = 2^M$ such that $\mu^2 \leq q \leq 2\mu^2$ [39]. Also, if gcd $(\zeta, \mu)$ is not equal to 1, then $\zeta$ is not a trivial factor of $\mu$. Otherwise, we proceed to the next task to find the order r which is the least integer satisfying $\zeta^r \equiv 1$ in modulo $\mu$. Now, considering a group consisting only of integers $Z$ in modulo $\mu$ (that is, $Z_\mu$) for constructing a function $F$: $Z_q \longrightarrow Z_\mu$, such that $F(\psi) = \zeta^\psi$ mod $\mu$ so that $F(\psi + r) = F(\psi)$ if $\psi + r \leq q$. Quantum Fourier transformation is the implementation of the Discrete Fourier transform for quantum purposes, which is used here to find the order $r$. Using equation 14 on the state $|0\rangle$, we will get:

$$\sum_{\psi=0}^{q} |x\rangle \tag{15}$$

Next, using DFT and applying a unitary transformation:

$$U_F: |\psi_1\rangle|\psi_2\rangle \rightarrow: |\psi_1\rangle|\psi_2 + \zeta^{\psi_1} mod\mu\rangle;\ \psi_1 \in \mathrm{Z}_q and \psi_2 \in \mathrm{Z}_\mu \tag{16}$$

That is very efficiently computable [39], $r$ can be found out by measuring register 2 and keeping the corresponding state $\sum_\Gamma |\psi_0 + \Gamma r \ mod \ q|$ where $\psi_0 \in Z_r$ and repeating the process. If $r$ is an odd integer, then start from the initial step; otherwise, go for the factor $(\zeta^{r/2} + 1) \not\equiv 0$ mod μ and $(\zeta^{r/2} - 1) \not\equiv 0$ mod $\mu$, where $(\zeta^{r/2} + 1)(\zeta^{r/2} - 1) = \zeta^r - 1 \not\equiv 0$. Then find one of the factors $d_1 = \gcd(\zeta^{r/2} - 1, \mu)$ and consequently another factor $d_2$ [40].

## 5. Economy, Economics, and Business Cycles

Development in the finance sector plays a significant role in economic growth. There is a strong relationship between business cycles, economics, and the economy. Many authors [41, 42] have introduced the Fourier series and the Fourier transform, two of the econometric analysis techniques [43] used in Economics and the study of business cycles. The contribution of the share market is significant to economic growth. In this regard, many researchers have studied the share indices of countries such as Australia [43], New York [44], and India. In the following section, we see that the Fourier transform is used in spectral analysis to analyze and predict a country's business cycle.

## 6. Signal or Image Processing

One-dimensional continuous FT of the function $\eta$: R(real numbers)→ C (complex numbers) is symbolized as H(ϑ) and stated by:

$$H(\vartheta) = \int_{-\infty}^{\infty} \eta(\theta)e^{-2\pi i\theta\vartheta}d\theta \quad (17)$$

And the inverse Fourier transform is:

$$\eta(\theta) = \int_{-\infty}^{\infty} H(\vartheta)e^{2\pi i\theta\vartheta}d\vartheta \quad (18)$$

Also, DFT (Discrete Fourier transformation) is obtained from equation 12. Two-dimensional continuous Fourier transformation of the function $\eta(\theta, \vartheta)$ over the $\theta\vartheta$ plane is denoted by $H(\sigma, \varsigma)$ and defined by:

$$H(\sigma,\varsigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta(\theta,\vartheta)e^{-2\pi i(\sigma\theta+\varsigma\vartheta)}\, d\theta d\vartheta \quad (19)$$

Its inverse Fourier transform is:

$$\eta(\theta,\vartheta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\sigma,\varsigma)e^{2\pi i(\sigma\theta+\varsigma\vartheta)}\, d\sigma d\varsigma \quad (20)$$

Where $\sigma$ and $\varsigma$ are spatial frequencies (Number of repeated occurrences per unit distance, whereas that is per unit time for normal frequency. In S.I. units, spatial frequency is the number of cycles per unit meter) It is used to improve the normalization of fingerprint frequency [45]. In two dimensions, Discrete FT (DFT) is:

$$\boldsymbol{D}(\sigma,\varsigma) = \frac{1}{m'n'}\sum_{\theta=0}^{m'-1}\sum_{\vartheta=0}^{n'-1} \boldsymbol{B}(\theta,\vartheta)e^{-2\pi i\left(\frac{\sigma\theta}{m'}+\frac{\varsigma\vartheta}{n'}\right)} \quad (21)$$

Where variable frequency of frequency domain, σ and ς takes any value from 0, …, $N' - 1$. Its reverse is:

$$\boldsymbol{B}(\theta,\vartheta) = \frac{1}{m'n'}\sum_{\sigma=0}^{m'-1}\sum_{\varsigma=0}^{n'-1} \boldsymbol{D}(\sigma,\varsigma)e^{2\pi i\left(\frac{\sigma\theta}{m'}+\frac{\varsigma\vartheta}{n'}\right)} \quad (22)$$

Variable θ and ϑ of the spatial domain take values from 0, …, $N' - 1$; amplitude (D(σ, ς) amp) and phase ($\boldsymbol{D}(\sigma, \varsigma)$ phase) of the DFT $\boldsymbol{D}(\sigma, \varsigma)$ are [46]:

$$\boldsymbol{D}(\sigma,\varsigma)_{amp} = \sqrt{\Re^2(\sigma,\varsigma) + \Im^2(\sigma,\varsigma)};\ \boldsymbol{D}(\sigma,\varsigma)_{phase} = tan^{-1}\left[\frac{\Im(\sigma,\varsigma)}{\Re(\sigma,\varsigma)}\right] \quad (23)$$

It is very useful in signal processing [47] and is also used to analyze the morphology of coral reefs and karst [48].

The number of required operations to compute DFT using the standard two-dimensional Fast Fourier Transform (FTT) algorithm [49] is $N^2\log_2 N$ for complex multiplication and $2N^2\log_2 N$ for complex addition if the sample size is $N$, and it is 3/4 $N^2\log_2 N$ for complex multiplication and $2N^2\log_2 N$ for complex addition [50] using two-dimensional FFT by analogy of Cooley-Tukey algorithm [34]. The Fourier transform has some beautiful properties [51] that are

1) It works as a linear operator. If "$h(\xi) = ah_1(\xi) + bh2(\xi)$", and "$H(\Xi)$, $H_1(\Xi)$ , $H_2(\Xi)$" are the corresponding FT of the functions $h(\xi)$, $h_1(\xi)$ and $h_2(\xi)$ then "$H(\Xi) = aH_1(\Xi) + bH_2(\Xi)$"; using the equation 5.

2) In the time domain, if the function $h(t)$ is expanded by a certain amount, then its Fourier transform is compressed in frequency by the same amount. Therefore, the Fourier transform follows the time scaling.

Let $h(t) = h_1(t/a)$ and FT of $h(t)$, is $H(\chi)$ while that of $h_1(t)$ is $H_1(\chi)$, then:

$$H(\chi) = \int_{-\infty}^{\infty} h_1\left(\frac{t}{a}\right) e^{-2\pi i\chi t} dt \tag{24}$$

Let $u = t/a \Rightarrow t = ua$ then $dt = a.du$ so:

$$H(\chi) = \int_{-\infty}^{\infty} h_1(u) e^{-2\pi i\chi au}(a.du) = a\int_{-\infty}^{\infty} h_1(u) e^{-2\pi i\chi au} du = aH_1(a\chi) \tag{25}$$

Suppose an audio signal is played in 10 seconds, which is 3 times compressed of the original signal of 30 seconds with frequency 1500 Hz, then $h(30) = h_1(10)$, then $H(1500) = 3H_1(4500)$.

3 If $h(t) = h_1(t - a)$, $a(> 0)$ be a real number and Fourier transformation of $h(t)$ and $h_1(t)$ are $H(\chi)$ and $H_1(\chi)$ respectively then we get as previous way:

$$H(\chi) = e^{-2\pi i\chi a} H_1(\chi) \tag{26}$$

Hence:

$$h_1(t - a) \overset{\boldsymbol{F}}{\leftrightarrow} e^{-2\pi i\chi a} H_1(\chi) \tag{27}$$

Therefore, if a function is delayed by time, then its Fourier transformation will be multiplied by a complex exponential whose magnitude is 1.

4 Let three functions "$\eta_1(t)$, $\eta_2(t)$ and $\eta_3(t)$" such that "$\eta_1(t) = \eta_2(t)*\eta_3(t)$" is called convolution and also if Fourier transform of "$\eta_1(t)$, $\eta_2(t)$ and $\eta_3(t)$ are $H'(\chi)$, $H'_1(\chi)$ and $H'_2(\chi)$" respectively then "$H'(\chi) = H'_1(\chi)*H'_2(\chi)$". Therefore, "two functions' combined Fourier transform is equal to the sum of their individual Fourier transforms". Convolution is the third signal of mixing two signals. In digital signal processing, this technique has a major role.

In continuation of the properties of the Fourier transformation, the importance lies particularly in the convolution properties that have been emphasized in this context. There are reports that when two signals are merged, "the Fourier transform of the convolution of those two signals is the FT of the third signal, which is the output signal merged by the two signals" [52].

When we capture photos that contain a phase of light or when we record any music or sound, sometimes there is some unwanted data that is captured or recorded, called noise, that is stuck in the background of the photos or sound. We use advanced professional software to remove or avoid that noise or signal, resulting in better-quality photos or sound. This method is a signal-processing technique that isolates the signal to obtain more useful information. Signal is a function of time that carries information of the event occurring at that time [53], and it may also be defined as the changes in observation in quality as well as quantity [54]. Signal processing is the analysis, manipulation, and reconstruction of signals and helps to separate information from noise. There are different components of modern signal processing, including detection, localization, modification, classification, and tracking. Since Fourier transforms convert the time domain to the frequency domain, they play an important role in signal processing and can be used in filtering, modulation and demodulation, sampling techniques, and many other fields [55].

In the industrial field, the Fast Fourier Transform (FFT) is an important tool for identifying and resolving faults and errors in induction motors. The output signal of an induction motor in different situations, like a broken end connector ring, a broken bar, components of an induction motor, and the normal condition, has been characterized by FFT. In 2019, Yoo identified the instrumental construction fault or component defect, like "one broken bar, two broken bars, and three broken bar systems," through FFT of the current signal [56].

"The fractional Fourier transform (FRFT)" is a collection of linear transformations that generalizes the Fourier transform in the field of harmonic analysis in mathematics. A function can be transformed to any domain between frequency and time because n does not have to be an integer. It is comparable to the Fourier transform raised to the nth power.

Fractional FT of the function ⊔(t) is $F_\gamma[⊔](u)$, then:

$$F_\gamma[\sqcup](u) = \sqrt{\frac{1 - i\cot\gamma}{2\pi}}\, e^{\pi i \cot(\gamma) u^2} \int_{-\infty}^{\infty} e^{-2\pi i}\left(cosec(\gamma)ux - \frac{\cot\gamma}{2}x^2\right) f(\sqcup)dt \quad (28)$$

where $\gamma$ is the rotation angle, shown in Figure 6.

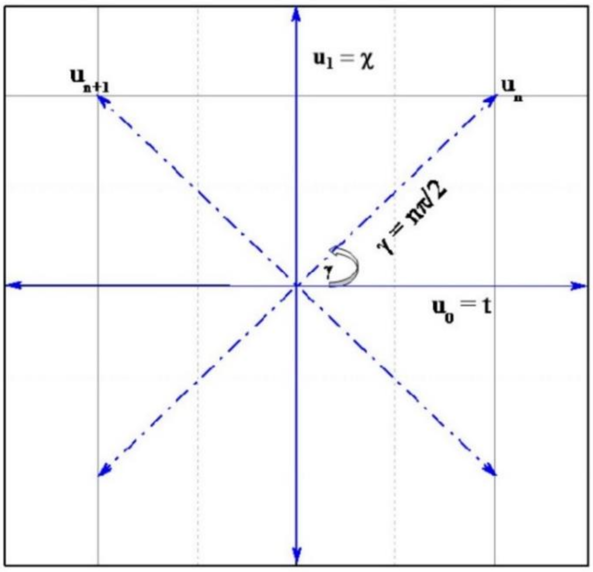


**Figure 6.** The FRFT domain is represented by the SFRFT frame, denoted by dotted lines (*un* and *un+1* axes, and they are orthogonal). $\gamma$ is the orientation angle with respect to the SFT frame, denoted by a continuous line (*u0* and *u1* are orthogonal axes. *u0* and *u1*represent the time and frequency axes, respectively). A signal is modeled in a plane with orthogonal axes for the frequency and time domains in TF (time-frequency) analysis. A signal's typical FT $F(u)$, which corresponds to the frequency axis for signals with $f(t)$ along the time axis, causes the TF plane to rotate by 90° as a result of the FT operation. The $F\gamma$ operator's FRFT of real order n allows for a middle representation between the time and frequency domains.

Applications include phase retrieval, signal analysis, filter construction, the discrete Fourier transform [57], and pattern recognition [58, 59], which are used as tools in signal processing. In the early 1990s, the Fractional Fourier transformation [59] of the function f(x) was introduced. The fractional Fourier transform (FRFT) offers many advantages over the

Fourier transform and can serve as an important tool in signal processing [58]. With respect to asymptotic properties of the fractional Fourier cosine and sine transform, Maksimovic *et al.* (2024) showed a "Tauberian type result" that intimately relates the "quasi-asymptotic" behavior of both even and odd distributions [59].

The "Chip-Fourier transform" (CFT) is the most important technique used in signal processing. These are crucial tools for ultrasonic imaging and parameter estimation. There are several fast algorithms for uniform CFT, but Sun and Qian (2024) have proposed a fast algorithm for non-uniform CFT for the best result with time complexity $O(N\log N + N\log(1/\epsilon))$. Security of protection of any image nowadays is very important [60]. To keep an image protected, encryption is required. For this purpose, the discrete Fourier transform (DFT) and the fractional Fourier transform (FRFT) have been widely used. Using DFT and FRFT with a double random-phase matrix, the proposed scheme has encrypted and decrypted an image, and its security has been analyzed using the Discrete Fractional Fourier transform [61 - 63]. Several researchers have recently begun looking into the FRFT of laser beams [64-74] and examining "the propagation characteristics of a vortex Hermite-cosh-Gaussian beam (vHChGB)" [75] moving through an optical system using "the fractional Fourier transform (FRFT)" [76]. Image reconstruction can be performed by the FT using the gridding method [77], and hybrid encryption can be achieved using the FRFT [78].

## 7. Operations Research

Operations research [79] is the subject that helps us develop better strategies for decision-making problems and provides optimal solutions using advanced analytical methods. In any business or industry, whether small or large, whether by oneself or shareholders, the primary aim is to maximize profit. The role of management is very important in achieving this optimal result. Here, in industrial or business management, "Supply Chain Management (SCM)" [80], a major part of operations research, plays a significant role in improving business quality by enhancing the stability of relationships among members of the supply chain. Many experts have analyzed supply chain management with the help of FT [81-84]. Without the stability of SCM, the inter-enterprise relationship (upstream-downstream) within the supply chain node will not be healthy. Stability of SCM plays a key role in the effective and healthy operation of the system. The stability of SCM may be broken down by internal shock load as well as external shock load. But the damage from the external shock is so much that sometimes it breaks the whole system. In this regard, Xu *et al.* (2021) have introduced the Flexibility Dynamics Vibration Model, where the wave function has been provided in terms of a Fourier transform to analyze stability. Here, the shock load wave and inherent wave are superimposed, and the data are expressed using DFT, FFT, and IFFT in the digital signal processing toolbox [84]. Actually, FFT is used as a tool to solve DFT efficiently. DFT can be solved in $O(N^2)$ time, containing n data entries, while FFT takes O(Nlog2N), which helps tremendously in completing a task in linear time [50].

Another type of problem in SCM has been studied using the Fourier transform, namely 'single manufacturer and multi-retailers with discrete stochastic demand'. In this problem, FFT has an important role in reducing the time complexity to resolve "the integer programming problem [85, 86]". In this case, if demand by retailers is not greater than the manufacturer's capacity, then it is trivial, but if it is not, then the problem arises. Considering both cases, the profit equations for retailers and manufacturers have been constructed, as well as for Make to Order and Make to Stock [87]. The game model is converted into an integer programming problem, which is usually solved by Dynamic programming [88,89]. If the manufacturer's production capacity is C′ and the number of retailers is $m_r$, then the time complexity using dynamic programming is $O(m_r C'^2)$. As the number of retailers and manufacturer capacity increase, it becomes more difficult to solve using the dynamic programming method, and the computational complexity grows larger. But using FFT is resolvable in $O(m_r C')$, that is, the linear time scale [84]. This is the most significant reduction in time complexity for solving an integer programming problem using FFT.

## 8. Technological Developments in FTIR Spectroscopy and Fantastic Versatile Applications

The reflectance mode of FTIR spectroscopy clearly reveals the surface morphology and the internal structure of semiconductor, dielectric, and polymer films. The source beam can be used to operate in single- or multiple-bounce ATR spectroscopy with a specular reflectance configuration. For the purpose of a Specular Reflectance Attachment (SRA), an accessory is essential in addition to the Regular Transmission Mode of Spectroscopy. Due to multiple internal reflections, an optical distortion arises, which is eliminated by applying the Kramers-Kronig correction in relevant software. A fully functional workflow is automatically generated, along with high-speed processing enabled for the obtained spectrum. Subsequently, by applying the Kramers-Kronig correction to diffuse reflectance measurements during optical processing, spectral aberrations can be eliminated, greatly enhancing the optical data library. In essence, the spectra from the region of interest are processed using a Kramers-Kronig transformation, which converts the derivative-shaped peaks caused by strong specular reflection into spectra that can then be cross-referenced against spectral libraries to identify the necessary component.

The Fourier spectral collection approach was used to discretize the space variable in the numerical study of the “generalized Rosenau-Korteweg-de Vries-regularized long wave (GR–KDV–RLW) equation”, whereas the central finite difference method was employed for the reliance on time [85]. FTIR Spectroscopy has also proved vital to analyze large area precious samples. In recent developments, very interesting applications involved a big mineral sample, an electrochemical cell, an inert atmosphere cell, and a heating and cooling stage in place, a Nicolet FTIR Microscope stage.

Very Fast Processing FTIR microscopy has proved vital in the pharmaceutical industry, supporting product development, quality control, and biologics formulation [90, 91]. FTIR analysis has entered the zone for conservation and restoration of Fine Arts and Historical objects or entities. External – Reflectance “Attenuated Total Reflection (ATR)” and “Fourier Transform Infrared (ER-FTIR)”, “ATR-FTIR (Fourier Transform Infrared) Spectroscopy”, “Raman Spectroscopy”, and “Fibre Optics Reflectance Spectroscopy (FORS)” have proven to be remarkable complementary analytical tools - “GC–MS (Gas Chromatography – Mass Spectroscopy)” for state-of-conservation, understanding processes of degradation, restoration of painting materials, and techniques of artwork, bringing in a revolution in Heritage Science and Modern Oil – Painting [92], Fine Arts & Historical Objects [93-95], and Analysis of commercial pigments and composition in Paintings [96].

In recent times, there have been reports on ATR-FTIR spectroscopy and spectroscopic imaging processes that have been employed for the analysis and production of biopharmaceuticals. These are drugs derived from biological sources for the treatment of diseases such as arthritis, diabetes, and certain types of cancer. Moreover, it is also being viewed as potentially capable of producing monoclonal antibodies to support the ongoing fight against SARS-CoV-2 [90].

"Fourier transform Infrared Spectroscopy (FTIR)" has widely in used for medical science to detect the various deadly disease like different types of cancer [97-106] and in different sectors like Glycosylation monitoring in living cells [107], examine the rat organs' varying biochemical profiles [108], distinguish between glioma tissues and regular brain tissues is demonstrated by peritumoral tissue of gliomas [109], examination of drug-carrier interactions [110], estimating the PEGylation level in biomolecules [111].

Artificial Intelligence (AI) [112] is revolutionizing many facets of contemporary life, including healthcare, entertainment, transportation, and education. Numerous advantages of the technology include greater accuracy, efficiency, and personalization. A hybrid forecasting model has been built up using DFT and an artificial neural network (ANN) [113] to improve forecast precision [114]. An important factor in the final cider's organoleptic properties is the apple cultivar utilized. In this case, classification of different apple varieties is important; ANN and FTIR provide a simple analytical strategy to address this problem [115]. Also, in the presence of AI and FTIR, it is possible to analyze the deterioration of lubricant oils caused by water [116].

## 9. Biomedical and Clinical Applications of Fourier Transform Infrared Spectroscopy

Fourier analysis, most notably implemented through FTIR spectroscopy, has become an indispensable analytical tool in biomedical, clinical, and forensic sciences. By transforming complex time-domain or absorption data into the frequency domain, Fourier analysis enables precise interpretation of molecular vibrations associated with specific functional groups in biological macromolecules. This synergy between mathematics and spectroscopy enables researchers to probe the biochemical composition of cells, tissues, and body fluids with remarkable sensitivity and specificity [117]. In particular, the "Attenuated Total Reflection-Fourier Transform Infrared (ATR-FTIR)" modality facilitates direct measurement of biological specimens with minimal preparation, making it highly suitable for in vivo and ex vivo medical applications.

In the medical field, FTIR spectroscopy has been widely used to elucidate biochemical fingerprints associated with disease states, tissue degeneration, and metabolic disorders. For instance, the spectral profiles derived from FTIR imaging have been used to identify alterations in protein secondary structures, lipid peroxidation, and nucleic acid conformations that occur during cancer progression and neurodegenerative diseases such as Alzheimer's [118, 119]. The technique provides a non-destructive, label-free alternative to conventional biochemical assays, enabling clinicians and researchers to monitor cellular metabolism and disease biomarkers in real time. Additionally, FTIR-based Fourier analysis has been successfully integrated into clinical diagnostics to detect β-amyloid (Aβ) fibril formation, evaluate cartilage and bone tissue integrity, and analyze blood plasma for metabolic shifts indicative of cardiovascular or pulmonary disorders.

Beyond diagnostic applications, Fourier-based spectral data are increasingly being combined with chemometric algorithms and machine learning frameworks to enhance pattern recognition and classification performance. Advanced multivariate methods, such as principal component analysis (PCA) and partial least squares discriminant analysis (PLS-DA), enable the extraction of diagnostically relevant spectral features, thereby improving the sensitivity and specificity of disease detection [117, 120]. In the context of personalized medicine, such integrative approaches have facilitated the identification of subtle biochemical differences between healthy and diseased tissues, enabling the development of predictive models for early disease detection. The fusion of Fourier spectral analysis with artificial intelligence thus represents a major step toward automating and standardizing spectroscopic diagnostics.

Overall, Fourier analysis through FTIR and ATR-FTIR spectroscopy offers a rapid, non-invasive, and highly reproducible method for analyzing the biochemical landscape of biological systems. Its capability to detect molecular alterations at the microstructural level makes it an invaluable tool in medical diagnostics, pharmacological monitoring, and forensic investigations. As technological advancements continue to improve spectral resolution, signal processing, and data analytics, Fourier-based methodologies are poised to play an even greater role in clinical decision-making, therapeutic monitoring, and biomarker discovery across the medical sciences.

## 10. Versatile Roles of Fourier Transform Spectroscopy in Modern Science

There are further uses for FT besides those mentioned above. We highlight here those areas. Inexpensive FT-NIR (Fourier Transform Near-Infrared Spectrometry) [121] provides the ability to measure and analyze different parameters of any sample without using any extra chemicals. So it is used very safely to get emission/absorption in “the infrared spectrum” of any one of “solid, liquid, or gas more precisely. It takes less time to provide results. It provides a spectrum from the raw sample or data using a Fourier transform. Also, there are many other spectroscopes where the “Fourier transform” has been deployed. Different fields of science have benefited from using this device. In the gas processing to remove $CO_2$ and other gas acids by concentrating amine solvent, FT-NIR helps to determine these components [122-127]. It helps to analyze different types of oil, like natural oil [128-131], bio-oil [132,133], extraction from the leaves [134], seeds [135], and analysis of quality, fat, and oil in the food products [136-139]. It is also employed to identify chemicals in microplastics [140] and analyze oil-dissolved gas in power transformers [141]. FT-NIR is particularly useful for studying Trigonella corniculata and Spinacia oleracea in naturally polluted soil with biochar addition. [142] and used to test for any harmful metal contamination in water and solids [143], and also useful for estimating quality parameters in oil from Indian mustard [144]. There are many techniques to test for drug abuse. Among them, hair analysis is very remarkable for its methodology and the abundance of samples. This test is also done by FT-NIR spectroscopy [145]. Many criminal cases often happen, like bike and car accidents, and run-and-hit accidents. To identify a victim’s bike or car paint, examination is a very efficient technique in a forensic laboratory, where paint is used as evidence of such accidental events. Raman spectroscopy is here the main tool to trace out the paint [146]. Identification of body fluids plays a key role in criminal investigation and forensic analysis. This identification process is also possible with the help of Fourier transform infrared spectroscopy [147]. Empirical research of connective tissues, including classification of ligament types and non-destructive evaluation of the growth of synthetic cartilage by near-infrared spectroscopy [148,149]. FTNIR spectroscopy is also used as an important tool for “in vivo” analysis of human muscle [150], to extract the chemical properties during oxidation of oil [151], to determine alterations in the chemical makeup of engine oils [152], to study the condition of used engine oil for reuse [153], and to calculate the overall acid number in mineral oils used in aviation engines [154]. Fourier transform micro spectroscopy [FTMS] has been used as an important tool for testing the chemical makeup of biological samples by analyzing their vibration motions [155], used to monitor for adulteration in food products [156], and used to evaluate the sorption of water in natural fibers on a qualitative and quantitative level [157]. Also, using vibrational spectroscopy [158], it has been shown that phenol's photochemical reaction occurs very quickly at the air-water interface [159].

## 10. Conclusions

In our present review work, we have highlighted that the Fourier transform (FT) and its subsequent developments, such as FFT, DFT, and FRFT, were not possible without the concept of the Fourier series, whose inventor, Joseph Fourier, was a great mathematician of all time. In connection with finding "The solution of the heat equation" in 1822, Fourier introduced, in his groundbreaking work "The Analytical Theory of Heat," the first infinite series of sums of sine and cosine functions, later known as the Fourier series. It is in the bone marrow of scientific progress, helping almost every branch of science, including cryptography, quantum cryptography, image and signal processing, operations research, biological sciences, and many areas of forensic research. We observe that it is none other than the Fourier series (FS) that enables the construction of an efficient algorithm for encryption and decryption, thereby providing a masking function for a symmetric-key cryptosystem. Using the Fourier transform (FT) scheme, the primary objective has been to enhance the secrecy of data transfer. It also plays a vital role in asymmetric key cryptosystems and is highly responsible for the inverse Fourier transform (IFT) for the protection of biometric signatures. There has been tremendous development, growing by leaps and bounds almost every day, in applying FT across various fields. As a consequence, we get the result also for the discrete case. DFT and FFT are the direct consequences and are used as a central part of quantum cryptography. Actually, FFT is a branch of instructions and is used as a tool to compute DFT. This is a significant algorithm that was discovered due to its low computational complexity. We have the Quantum Fourier Transformation (QFT) and Shor's algorithm, which have been implemented using DFT and FFT. This result calls into question the prevailing view of classical cryptography and highlights its limitations. This shows that the Fourier transform is highly effective in developing technology to protect the privacy of our data in the modern digital lifestyle. Besides its digital technology purposes, it also helps with economic forecasting and analyzing GDP values to achieve optimal output. Further, DFT and FFT help us to solve many problems that arise in the supply chain management of operations research. Stability analysis of SCM can be done by the DFT and FFT toolbox. FFT is used to reduce time complexity, maximizing the profit matrix in SCM. Basically, the "Fourier transform (FT)" converts "the real-time domain to the frequency domain," which helps us mostly in signal processing and image processing. A variety of processing area technologies have been enriched and achieved by FT. Many researchers also developed FT to explore it. In early 1990, a new form and alternative interpretation of FT was introduced, that is, FRFT. This is the most recent development, which has helped with image encryption in many situations, such as satellite imagery and image privacy protection. Building on the profound idea of FT, various spectroscopic techniques have been employed in different research works across biological and chemical sciences, as well as in critical criminal investigations involving forensic laboratories. Advanced studies on DFT, FRFT, and the combined discrete fractional Fourier transform have led to powerful applications. Not only will Fourier be recognized for his exceptional discovery, but also for his discretion in successfully leading an administrative role and his positivism in social activity.

## Author Contributions

Conceptualization, K.R.S., S.K.D. and A.D.; methodology, K.R.S., S.K.D. and A.D.; software, K.R.S. and S.K.D.; formal analysis, K.R.S., S.K.D. and A.D.; writing—original draft preparation, K.R.S., S.K.D. and A.D.; writing—review and editing, U.K.B. and B.S.; supervision, K.R.S. and A.D. All authors have read and agreed to the published version of the manuscript.

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## Conflicts of Interest

The authors declare no conflict of interest.

## Abbreviations

The following abbreviations are used in this manuscript:

| Abbreviation | Definition |
|---|---|
| $l$ | Length of the String |
| $y(x, t)$ | Represent the Function of x and Time t Both |
| $y_{tt}$ | “2nd Partial Derivative of the Function y with Respect to t” |
| $y_{xx}$ | “2nd Partial Derivative of the Function y with Respect to x” |
| $c$ | Speed of Light in S. I. Unit |
| $u$ | Temperature Distribution Function |
| $\mathbf{r}$ | Position Vector in 3D Space |
| $l$ | Length of the String |
| $\kappa$ | Constant of Proportion |
| $k$ | Heat Conductivity |
| $\sigma$ | Specific Heat |
| $\rho$ | Density |
| $\chi$ | Frequency |
| $\hat{f}(\omega)$ | Function of Frequency |

| Abbreviation | Definition |
|---|---|
| $\mathcal{F}(\eta(t))$ | Fourier Transformation of a Function $\eta(t)$ |
| $b_1$ | Plain Text |
| $b_2$ and $b_3$ | Two Biometric Signatures |
| $c_2$ and $c_3$ | Biometric Trait Images ( as Fingerprint and Iris) |
| $\langle X\|Y\rangle$ | Represent the Quantum State that is pronounced by Bra and Ket |
| $\mu$ | Product of Two Primes u and v; it is the Public Key |
| $\phi(\mu)$ | Prime Positive Integers to μ that are Less than μ |
| N mod $\phi(\mu)$ | Remainder After Divided by ϕ(μ) |
| $d'$ = N mod $\phi(\mu)$ | It is a Private Key |
| $N'$ | Number of Data Sets in the Vectors |
| $O(N'^2)$ | Represents Big O Notation that is Upper Limit; Here $N'^2$ is the Maximum Value |
| $U_F$ | Unitary Transformation |
| $H(\vartheta)$ | One-Dimensional Continuous Fourier Transformation |
| $\eta$ | One-Dimensional Inverse Fourier Transformation |
| $H(\sigma, \varsigma)$ | Two-Dimensional Continuous Fourier Transformation |
| $\eta(\vartheta, \theta)$ | Two Dimensional Inverse Fourier Transformation |
| $\mathbf{D}(\sigma, \varsigma)$ | Discrete Fourier Transformation |
| $\mathbf{D}(\vartheta, \theta)$ | Discrete Inverse Fourier Transformation |
| $\Re(\sigma, \varsigma)$ | "Real Part of the Function" |
| $\Im(\sigma, \varsigma)$ | Imaginary Part of the Function" |

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## Supplementary Materials

### Appendix-A

"Consider a string with length $l'$ and tagged at two final points at A and B", shown in Figure 7. Frequency ($\chi$) equals the ratio of velocity ($v$) and wavelength ($\lambda$) ($\chi = v/\lambda$). Fundamental frequency [$\chi_r$; $r$ = 1, 2, 3] is the lowest value ($\chi_1$), generated due to an oscillating string with length $l$.

1) If $l = \lambda / 2$, $\chi_1$ is equal to $v / 2l$; it is also called the 1st harmonic.
2) If $l = \lambda$, $\chi_2$ is equal to $v / l = 2\omega'_1$; it is also called the 2nd harmonic.
3) If $l = 3\lambda / 2$, $\chi_3$ is equal to $3v / 2l = 3\omega'_1$; it is also called 3rd harmonic.

Hence, we obtain that $\chi_n = n\chi_1$; It is also called the $n$th harmonic.

### Appendix-B

"Fourier series of the function $\eta(x)$ stated on a finite domain $[-s, s]$" can be expressed as follows:

$$\eta(x) = \sum_{k=-\infty}^{\infty} c_k \left(\cos\frac{k\pi x}{s} + i\sin\frac{k\pi x}{s}\right) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi x}{s}} \tag{29}$$

Where the Fourier coefficients $c_k$ are given by:

$$c_k = \frac{1}{2s}\int_{-s}^{s} \eta(x)\, e^{-\frac{k\pi i x}{s}}\, dx \tag{30}$$

And frequency:

$$\chi = \frac{k\pi}{s} = k\Delta\chi \tag{31}$$

This shows that η(x) is totally periodic throughout the interval. To extend this function to infinite length, we choose s tends to infinity, then Δχ tends to zero. Therefore, we have used equations 29, 30, and 31

$$\eta(x) = \lim_{\Delta\chi\to 0} \sum_{k=-\infty}^{\infty} \frac{\Delta\chi}{2\pi}\int_{-\frac{\pi}{\Delta\chi}}^{\frac{\pi}{\Delta\chi}} \eta(\tau)e^{-ik\Delta\chi\tau}\, d\tau\, e^{ik\Delta\chi x} \tag{32}$$

Using Riemann integration [160], we have:

$$\eta(x) = \int_{-\infty}^{\infty}\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\eta(\tau)e^{-i\chi\tau}d\tau\right]e^{i\chi x}d\chi \tag{33}$$

As $\Delta\chi$ becomes $d\chi$, writing $\chi$ as frequency for $k\Delta\chi$, we have:

$$\eta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}(\eta(x))\, e^{i\chi x}d\chi \tag{34}$$

where $\mathcal{F}\big(\eta(x)\big) = \int_{-\infty}^{\infty}\eta(\tau)e^{-i\chi\tau}d\tau$ is the Fourier transform of the function $\eta(\tau)$. Therefore, Fourier transform of the function $\eta(x)$ is:

$$\mathcal{F}(\eta(x)) = \int_{-\infty}^{\infty} \eta(x) e^{-i\chi x} dx \tag{35}$$

Also, similarly, we can write the inverse Fourier transformation from equation 34:

$$\eta(x) = \mathcal{F}^{-1}\left(\mathcal{F}(\eta(x))\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(\eta(x)) e^{i\chi x} d\chi \tag{36}$$

Using the Fourier coefficient, there are several applications through high power sums of holomorphic cusp forms' Fourier coefficients [161].

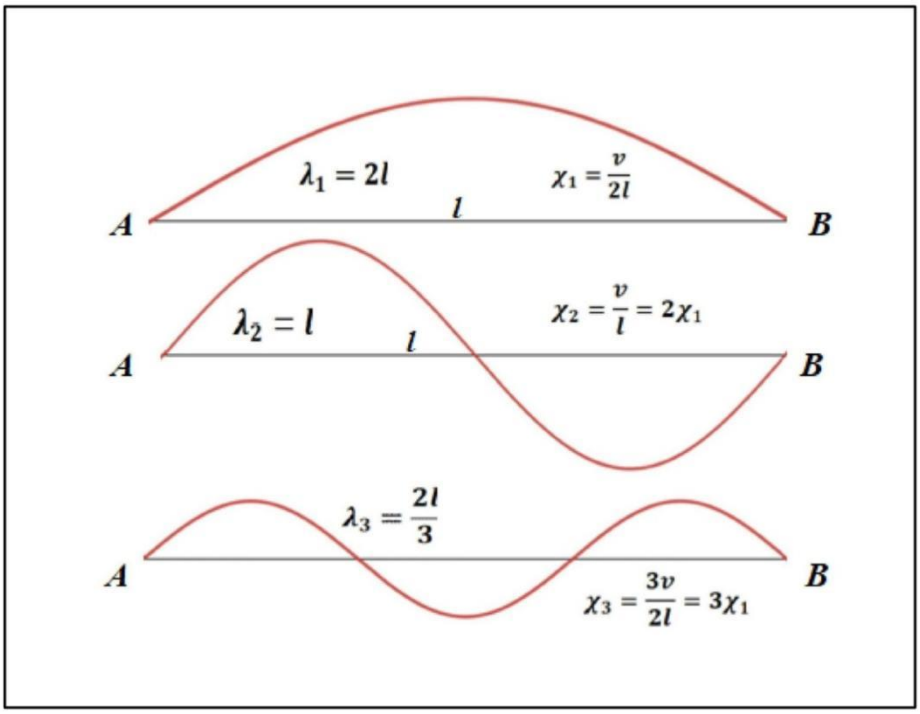


**Figure 7.** Fundamental frequency.

**Appendix-C**

If we consider now the discrete data set starting with $\eta_0$ at $x_0$, $\eta_1$ at $x_1$, and so on, ending with $\eta_{(n-1)}$ at $x_{(n-1)}$ as shown in Figure 8, then using discrete Fourier transformation, we have:

$$\hat{\eta}_k = \sum_{j=0}^{n-1} \eta_j e^{\frac{-2\pi ijk}{n}} \tag{37}$$

and inverse discrete Fourier transformation

$$\eta_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{\eta}_j \, e^{\frac{2\pi ijk}{n}} \tag{38}$$

So, $\{\boldsymbol{\eta_0}, \boldsymbol{\eta_1}, \boldsymbol{\eta_2}, \dots, \boldsymbol{\eta_{n-1}}\} \overset{\boldsymbol{DFT}}{\longleftrightarrow} \{\boldsymbol{\hat{\eta}_0}, \boldsymbol{\hat{\eta}_1}, \boldsymbol{\hat{\eta}_2}, \dots, \boldsymbol{\hat{\eta}_{n-1}}\}$ defined by the following Matrix relation:

$$\begin{bmatrix} \hat{\eta}_0 \\ \hat{\eta}_1 \\ \cdot \\ \cdot \\ \hat{\eta}_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdot\,\cdot & 1 \\ 1 & \chi_n & \chi_n^{(2)} & :\,: & \chi_n^{(n-1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ 1 & \chi_n^{(n-1)} & \chi_n^{2(n-1)} & \cdot\,\cdot & \chi_n^{(n-1)^2} \end{bmatrix} \cdot \begin{bmatrix} \eta_0 \\ \eta_1 \\ \cdot \\ \cdot \\ \eta_{n-1} \end{bmatrix} \tag{39}$$

Where $\chi_n = e^{-2\pi i/n}$.

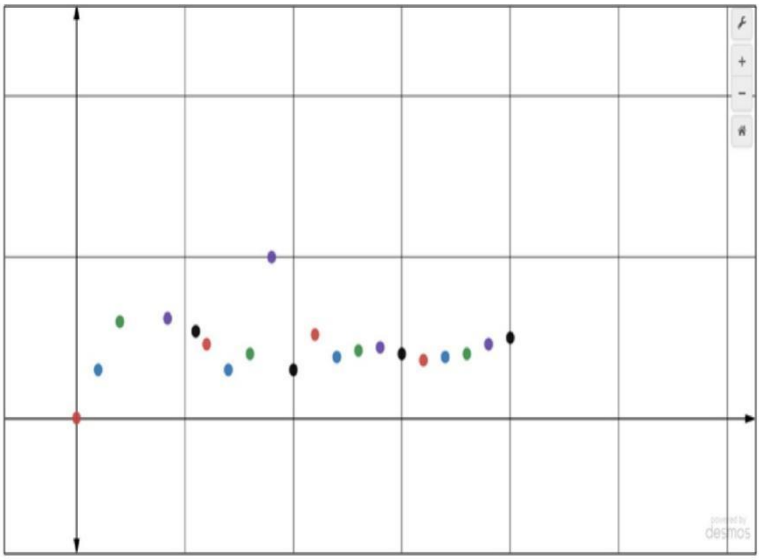

**Figure 8.** Discrete data points representation.